HABILITATION THESIS REVIEWER'S REPORT

Masaryk University

Applicant	John Bourke
Habilitation thesis	Categorical structures for higher-dimensional universal algebra
Reviewer	Ross Street, BSc PhD (Sydney), DSc (Macquarie), FAA (AustAcadSci)
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Review of the Habilitation thesis of John Bourke

The five published papers presented in this thesis represent a fine body of work written by a creative, careful and precise mathematician. The author is a master in the advancement of fruitful interactions amongst (i) 2-category theory, (ii) homotopy theory, and (iii) higher category theory. The needs of one of these theories can often come from one of the others but then that advance in the one can feed back to develop the other. I would say that Bourke's work is a major factor in turning these three theories into one beautiful union with applications in many fields.

I shall refer to the five papers of the collection as [11], [14], [13], [17], [23] in accord with the Bibliography of Chapter 1, Introduction, pages 21--28.

Paper [11] gives a completely fresh view on the 2-dimensional universal algebra required to cover the study of categories with structure (not just sets with structure). It was well understood that blind application of the theory of monads for 2-categories as categories enriched in the cartesian closed category Cat of categories did not deal with mathematical practice. The morphisms are too strict. There are three kinds of weaker morphism, each with its uses. Yet it was realised that weaker morphisms could be helped by the understanding of the strict; Lack and Shulman introduced a mechanism for that. Bourke's great insight in [11] was to discover how that mechanism gave the correct versions of doctrinal adjunction and monadicity theorems. The results in [11] has been applied by Gurski-Johnson-Osborn.

Algebraic weak factorization systems (awfs) were introduced by Garner to make the weak factorization systems, as used in Quillen's model categories, more manageable. Paper [14] gives a rundown of the theory up to that date and then shows how double categories naturally allow a characterization of awfs and a fundamental understanding of cofibrant generation. They provide many applications, even to non-algebraic cases. The work was further applied by Hess. In fact [14] is Bourke's most cited paper (16 citations registered today on Math. Reviews).

I totally agree with the referee of paper [13] who reported as follows:
This paper provides a vital step in understanding the symbiosis between category theory and homotopy theory. While skew monoidality is justified as a notable concept by its appearance in the study of quantum groupoids and related structures, this work clinches its importance much more generally.
The idea that a skew monoidal structure on a Quillen model category might pass to a genuine monoidal structure on the homotopy category is quite appealing and might have occurred to others. However, the genius of the author's analysis is to see that it is the skew closed structures that can be identified on important Quillen model categories arising in higher category theory and that these pass

to genuine monoidal closed structures on the homotopy categories. This breakthrough paper gives the appropriate abstraction of the process.
While the examples here (and in a future paper) come from higher category theory, the referee is convinced that the same phenomenon will be demonstrated in purely homotopical examples: that is, in homotopy algebras.
The homotopical version Theorem 5.12 of an Eilenberg-Kelly Theorem is a central result. While, to emphasise that the paper deepens understanding, the setting is shown to provide the conceptual solution sought by Hyland-Power for their result on pseudocommutative monads.
Indeed, the results have already been applied by Campbell.

The paper [17] with Garner reaches brings to a definitive point the study emerging on the monad side from algebraic topology and algebraic geometry (see the Appendix of the book on sheaves by Godement), and on the theory side from universal algebra (Garrett Birkhoff) and category theory (Lawvere). Lately the dichotomy was extended to include examples from higher category theory (Batanin, Berger, Melliès, Weber) through an understanding of nerve constructions.

Paper [23] is still to be formally accepted for publication but is available on the arXiv. It generalizes Freyd's Adjoint Functor Theorem in an innovative way to a powerful result applicable to higher category theory and Quillen homotopy model categories.

Here are a few typographical errors in the Introduction.

1. Page 4, line -8: <How about the about abstract approach?>
2. Page 5, line 3: <left> is a relic.
3. Page 13, line 6: Should it not be tensoring with the unit I that gives the monad and comonad?
4. Page 13, line -12: <is> should be <are>.
5. Page 13, line -9: insert <than>.
6. Page 11, 3rd last paragraph: <precisely> repeated.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

Question 1 Kelly invented *clubs* for aspects of 2-dimensional universal algebra. He saw, during 1971 lectures by Peter May, had close connection with the *operads* of homotopy theory. Where does this fit in your big picture relating 2-dimensional universal algebra and homotopy theory?

Question 2 On page 7 of the Introduction, you have a span (1.2.1) where the legs have special properties. Do you think there is a connection with the reviewer's work on polynomial functors as special spans?

Conclusion

The habilitation thesis entitled “Categorical structures for higher-dimensional universal algebra” by John Bourke **fulfils** requirements expected of a habilitation thesis in the field of Mathematics – Algebra and Theory of Numbers.

Date: 22 April 2021 Signature: